Sensitivity of quintessence perturbations to initial conditions

Jingsong Liu[1]

[1] *Dept. of Physics, Stanford University, Stanford, CA 94305-4060*

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Gauge invariant quintessence perturbations in the quintessence and cold dark matter ($QCDM$) model are investigated. For three cases of constant equation-of-state (EOS) parameter, linear scalar field potential, and supergravity scalar field potential, their perturbation evolutions have a similar dependence on EOS parameter and scale, but they have different sensitivity to the initial conditions due to the different shapes of the quintessence potential. They have a minor effect on primary CMB anisotropies, but change the secondary CMB effect in different ways. The first case is insensitive to initial quintessence perturbations and only modifies the Integrated Sachs-Wolfe (ISW) effect within a factor of 2. The other two cases are sensitive to initial conditions at large scales and could affect the secondary CMB anisotropies drastically depending on how smooth the initial perturbations are. This makes it possible for future cosmological probes to provide constraints on quintessence properties.

I. INTRODUCTION

Recent observations of an accelerating universe [1, 2, 3] imply the existence of dark energy characterized by a negative pressure to density ratio, also known as the EOS parameter w. There have been discussions of the cosmological constant, the dark energy evolving according to a specific EOS parameter [6, 7], and the dark energy consisting of a dynamical cosmic scalar field, the quintessence [10, 11]. In contrast to the cosmological constant, the quintessence component in $QCDM$ model has kinematic behavior and can develop perturbations. It has important features of time-evoluting EOS parameter and scale-dependent effective sound speed [13, 14]. One significant issue is the sensitivity to initial perturbation conditions, which attracts much attention.

For some $QCDM$ models, the quintessence potentials can be approximated with constant EOS parameter, and the perturbations evolutions are insensitive to the initial quintessence perturbations, with minor effect on observations [15, 16, 24]. For exponential potential $V(Q) = \hat{V}e^{-(c/M)Q}$ [8, 9, 10], although quintessence perturbations may stay non-zero at some time for different background attractor solutions, they die out in matter-dominant time for physically reasonable models [12], which shows the insensitivity again. For the tracking quintessence models [20], large-scale non-adiabatic perturbations can grow or stay constant before entering tracker regime, but have to get suppressed afterwards [21]. However, the first-order matrix formulation developed latter [22] applies to general quintessence potential cases on all linear scales and indicates the possibility of non-vanishing entropy perturbations. Recently some other $QCDM$ models with special quintessence potentials, such as linear $V(Q)$ [26] and supergravity $V(Q)$ [25], lead to special background evolutions. So it will be interesting to see the perturbations behavior and their sensitivity to initial conditions in these models, as well as their effects on cosmic microwave background (CMB) [4] and baryon mass power spectrum [5] in the universe.

In this paper, we describe the universe by a simple $QCDM$ model with the conformal-Newtonian metric in section 2. The cosmological perturbations are solved for constant w cases in section 3, for linear-$V(Q)$ cases in section 4, and for supergravity-$V(Q)$ cases in section 5. The results are analyzed by the matrix formulation and their effects on cosmological observations are discussed in section 6.

II. QUINTESSENCE AND COLD DARK MATTER (QCDM) UNIVERSE

Here we consider the uniform matter and quintessence field background with small perturbations in a perturbed flat FRW metric. Focusing on scalar perturbations, we can use the flat conformal-Newtonian metric for the gauge invariant approach [28, 29]:

$$ds^2 = a^2(\eta)\left[-(1+2\Phi(\eta,\vec{x}))d\eta^2 + (1-2\Phi(\eta,\vec{x}))\delta_{ij}dx^i dx^j\right] \tag{1}$$

where η is the conformal time and $|\Phi(\eta,\vec{x})| \ll 1$ is the gravitational potential. We define $' \equiv \frac{\partial}{\partial\eta}$, $\mathcal{H} \equiv \frac{a'}{a}$ as the "Hubble parameter" in the conformal-Newtonian version, η_0 as today's conformal time, and $\bar{a} \equiv \frac{a(\eta)}{a(\eta_0)}$ as the normalized scale factor. For the quintessence scalar field, we assume the standard Lagrangian $\mathcal{L}_Q = -\frac{1}{2}Q_{,\mu}Q^{,\mu} - V(Q)$ and decompose the quintessence field as uniform background with small spatial perturbations: $Q(\eta,\vec{x}) \equiv Q_0(\eta) + \delta Q(\eta,\vec{x})$, $|\delta Q| \ll |Q_0|$, $\langle\delta Q(\eta,\vec{x})\rangle = 0$, where $\langle\ \rangle$ means global spatial average. We assume that the quintessence part and matter

part only interact with each other gravitationally. So the wave equation of the quintessence field $-Q^{;\mu}_{;\mu} + \partial V/\partial Q = 0$ gives its background and perturbative components:

$$Q_0'' + 2\mathcal{H}Q_0' + a^2 V_{,Q}(Q_0) = 0 \;, \tag{2}$$
$$\delta Q'' + 2\mathcal{H}\delta Q' - \nabla^2\delta Q + a^2 V_{,QQ}\delta Q - 4Q_0'\Phi' + 2a^2 V_{,Q}\Phi = 0 \;. \tag{3}$$

The energy-momentum tensor of this quintessence field is $T_Q^{\mu\nu} = Q^{,\mu}Q^{,\nu} + (-\frac{1}{2}Q_{,\sigma}Q^{,\sigma} - V)\, g^{\mu\nu}$.

For the matter part, we assume its energy-momentum tensor has the form of the perfect fluid: $T^{\mu\nu} \equiv (\rho_m + p_m)U^\mu U^\nu + p_m\, g^{\mu\nu}$, where $U^\mu \equiv dx^\mu/d\tau = [(1-\Phi)/a, V^i/a]$ is the 4-velocity of the fluid, $V^i \equiv dx^i/d\eta = a\, dx^i/dt \ll 1$ is its 3-velocity, $\rho_m(\eta,\vec{x}) \equiv \langle\rho_m\rangle(\eta)[1+\delta_m(\eta,\vec{x})]$, and $|\delta_m(\eta,\vec{x})| \ll 1$. If we neglect all pressure effect ($|p_m(\eta,\vec{x})| \ll \rho_m(\eta,\vec{x})$, $|\nabla\delta p_m| \ll \rho_m|\nabla\Phi|$) on scales which are not too small (> 10 Mpc), the energy-momentum-conservation law gives background and perturbation components after some manipulation:

$$\langle\rho_m\rangle = \frac{\langle\rho_m\rangle(\eta_0)}{\bar{a}^3} \;, \tag{4}$$
$$\delta_m'' + \mathcal{H}\delta_m' - 3\Phi'' - 3\mathcal{H}\Phi' - \nabla^2\Phi = 0 \;. \tag{5}$$

The matter perturbation δ_m can be formally obtained in terms of a and $\Phi(k,\eta)$ in k-space, where k is the comoving wave-vector ($-\nabla^2_{\vec{x}} = k^2$).

To complete this system, we consider the 0th-order 0-0 component and the 1st-order $i{=}j$ component of the Einstein equations:

$$\frac{3\mathcal{H}^2}{a^2} = 8\pi G\left[\frac{\langle\rho_m\rangle(\eta_0)}{\bar{a}^3} + \frac{Q_0'^2}{2a^2} + V(Q_0)\right] \;, \tag{6}$$
$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2\left(-\frac{Q_0'^2}{a^2}\Phi + \frac{Q_0'\delta Q'}{a^2} - V_{,Q}\delta Q\right) \;. \tag{7}$$

Given the initial conditions, $V(Q)$, and other parameters, eq.(2, 6) give the background $Q_0(\eta)$ and $a(\eta)$, and eq.(3, 7) give the perturbations δQ and Φ. Due to the convenient conformal-Newtonian metric, the matter perturbation δ_m drops out and can be found later via eq.(5).

III. CONSTANT w CASE

Here we follow the EOS parameterization in [15] to study the general properties of the quintessence field instead of some specific quintessence field potential $V(Q)$.

$$w(\eta) \equiv \frac{\langle p_Q\rangle}{\langle\rho_Q\rangle} = \frac{\frac{Q_0'^2}{2a^2} - V(Q_0)}{\frac{Q_0'^2}{2a^2} + V(Q_0)} \;. \tag{8}$$

Then we have

$$\frac{Q_0'^2}{2a^2} = \frac{1+w}{1-w}V(Q_0) = \frac{1+w}{2}\langle\rho_Q\rangle. \tag{9}$$

By manipulating eq.(2, 9), we can express $V_{,Q}$ and $V_{,QQ}$ in terms of the EOS parameter $w(\eta)$:

$$a^2 V_{,Q} = -\frac{Q_0'}{2}\left[3(1-w)\mathcal{H} + \frac{w'}{1+w}\right] \;, \tag{10}$$
$$a^2 V_{,QQ} = -\frac{3}{2}(1-w)\left[\frac{a''}{a} - \mathcal{H}^2(\frac{7}{2} + \frac{3}{2}w)\right] + \frac{1}{1+w}\left[\frac{w'^2}{4(1+w)} - \frac{w''}{2} + w'\mathcal{H}(3w+2)\right] \;. \tag{11}$$

With eq.(10, 11) and definitions

$$Q_0' \equiv a(\eta_0)\sqrt{(1+w)\langle\rho_Q\rangle(\eta_0)}\;\psi', \quad \psi'(\eta_0) = 1, \tag{12}$$
$$\delta Q \equiv a(\eta_0)\sqrt{(1+w)\langle\rho_Q\rangle(\eta_0)}\;\delta\psi, \tag{13}$$

we can put the quintessence wave equations eq.(2, 3) in a more convenient form:

$$\psi'' + \frac{1+3w}{2}\mathcal{H}\,\psi' = 0, \tag{14}$$

$$\delta\psi'' + (2\mathcal{H} + \frac{w'}{1+w})\delta\psi' + \left\{-\nabla^2 - \frac{3}{2}(1-w)\left[\frac{a''}{a} - \mathcal{H}^2(\frac{7}{2}+\frac{3w}{2})\right] + 3\mathcal{H}w'\right\}\delta\psi$$
$$-4\psi'\Phi' - \left[3\mathcal{H}(1-w) + \frac{w'}{1+w}\right]\psi'\Phi = 0. \tag{15}$$

In this formulation, any function $V(Q)$ is equivalently described by the corresponding EOS parameter $w(\eta)$, which can be much more easily specified and compared with other dark energy models. One simple example would be $w = constant$, which has the exponential-like potential $V(Q)$ shown in Figure 1.

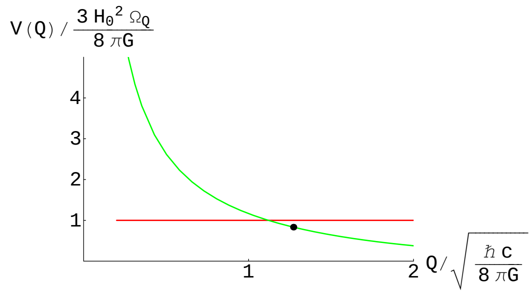

FIG. 1: Scalar field potential $V(Q)$ for constant-w cases. The curved line is for $w = -2/3$. The scalar field rolls from top-left corner down to the today's value indicated by the dot. The horizontal line is for $w = -1$ (ΛCDM).

In this special case, eq.(9, 12, 14) give the solution for the quintessence field background:

$$\psi' = \bar{a}^{-(1+3w)/2}\,, \tag{16}$$

$$\langle\rho_Q\rangle = \frac{\langle\rho_Q(\eta_0)\rangle}{\bar{a}^{3(1+w)}}\,. \tag{17}$$

Then the previous closed system of a, Q_0', Φ and δQ can be reduced to a system of a, Φ and $\delta\psi$ in k-space:

$$\frac{\mathcal{H}^2}{\bar{a}^2} = (H_0a_0)^2\left[\frac{\Omega_m}{\bar{a}^3} + \frac{\Omega_Q}{\bar{a}^{3(1+w)}}\right], \tag{18}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = \frac{3}{2}(H_0a_0)^2(1+w)\Omega_Q\left[-\frac{\Phi}{\bar{a}^{1+3w}} + \frac{\delta\psi'}{\bar{a}^{(1+3w)/2}} + \frac{3}{2}\mathcal{H}(1-w)\frac{\delta\psi}{\bar{a}^{(1+3w)/2}}\right], \tag{19}$$

$$\delta\psi'' + 2\mathcal{H}\delta\psi' + \left\{k^2 - \frac{3}{2}(1-w)\left[\frac{a''}{a} - \mathcal{H}^2(\frac{7}{2}+\frac{3w}{2})\right]\right\}\delta\psi - 4\frac{\Phi'}{\bar{a}^{(1+3w)2}} - 3\mathcal{H}(1-w)\frac{\Phi}{\bar{a}^{(1+3w)/2}} = 0, \tag{20}$$

where Ω_m and Ω_Q are the density parameters defined as usual. If $w = -1$, the quintessence perturbations vanish in eq.(19) and only its background contributes. This special case thus reduces to the ΛCDM model. At early times, this model can be well approximated by the flat CDM model (Einstein-DeSitter space). Then eq.(18) gives $\bar{a}(\eta) \propto \eta^2$, $\mathcal{H} \simeq \frac{2}{\eta}$, and eq.(19) gives Φ independent of time for the growing modes of matter perturbations $\delta_m \propto \bar{a}(\eta)$, as usual. At later times, the quintessence part begins to become important and the solution of $\bar{a}(\eta)$ from eq.(18) will deviate from η^2 more and more as time goes on. So the coefficient $2\mathcal{H}' + \mathcal{H}^2$ in eq.(19) is not zero any more and makes Φ decay. This is what causes the ISW effect in low-l multipoles of the CMB power spectrum.

Recent cosmological observation of the accelerating universe requires $w(\eta_0) < -1/3$. The $w < -1$ cases would correspond to negative kinetic term in the standard quintessence Lagrangian or other more complicated models, which we won't discuss here. So from now on we just consider $-1 \le w < -1/3$ in all our $QCDM$ models. The initial conditions are $a(0) \cong 0$, $|\Phi(0)| \sim 10^{-5}$ and $\Phi'(0) = 0$ for modes outside horizon, based on inflation theory [33]. For the same reason, $|\delta\psi(0)| \sim |\Phi(0)|$. Because in eq.(20) the restoring term ($V_{,QQ} \propto \mathcal{H}^2$) is bigger than the damping term ($\propto \mathcal{H}$) which is bigger than the driving term ($\propto \mathcal{H}\,\bar{a}^{-(1+3w)/2}$) at early times, $\delta\psi$ with any initial value which is not too big ($|\delta\psi(0)/\Phi(0)| \le 10^4$) will oscillate and get damped quickly without affecting the Φ evolution before the last-scattering epoch. So the primary effects on the CMB won't get modified much and are insensitive to the initial

condition on the quintessence perturbations. This is one important result within many $QCDM$ models [15, 16]. Here we can see that it is mainly due to the huge $V_{,QQ}$, i.e., the huge quintessence mass at early times. For now we just take the smooth initial condition $\delta\psi(0) = \delta\psi'(0) = 0$ for convenience. Let us look at the behavior of this system at various spatial scales.

At small scales ($k \gg 1$), by eq.(20) the growth of $\delta\psi$ is strongly suppressed and stays negligible compared with Φ. The effective sound speed of the quintessence perturbations approaches 1 ($c_s^2 \equiv \delta p_Q/\delta\rho_Q \simeq 1$). Then we can drop the $\delta\psi$ terms in the above system and the evolution of Φ only comes from background effect without a dependence on k in this region:

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = -\frac{3}{2}(H_0 a_0)^2(1+w)\frac{\Omega_Q}{\bar{a}^{1+3w}}\Phi \; .$$

This system behaves like the ΛCDM model with a different EOS parameter w and the additional $r.h.s$ term. If w is bigger the dark energy takes over dominance earlier and $\bar{a}$ deviates from η^2 earlier too, which makes Φ decay more. In addition, the negative $r.h.s$ brings down Φ more. So compared with the ΛCDM model, small-scale quintessence perturbations in the $QCDM$ model do not grow and only the background effect contributes, which drags down the gravitational potential more and corresponds to a bigger ISW effect.

At scales comparable to or larger than horizon size ($k \leq 1$), the quintessence perturbations $\delta\psi$ can grow to the order of the gravitational potential Φ by eq.(20) with smaller effective sound speed ($c_s^2 < 1$). Then it will back-react on the Φ evolution via eq.(19). Because of the positive coefficient, the quintessence perturbations tend to raise the gravitationally potential Φ to make it decay less. This is different from the $k \gg 1$ case, and agrees with the result of a phenomenological approach [23]. Thus in this $QCDM$ model the evolution of perturbations and gravitational potential depends on scale.

With definition on horizon scale $a_0 \equiv \frac{2}{H_0}$, numerical calculation gives $\Phi(\eta)$ and $\delta\psi(\eta)$ once we specify the parameters k and w. Corresponding to the assumptions we made before for this linearized system, we only consider $k < 800$. For $w = -0.5$ and $k = \{0.01, 5, 500\}$, the time evolution $\Phi(\eta)$ is shown compared with the $w = -1$ case in Figure 2.

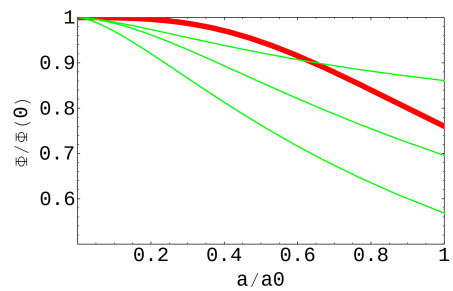

FIG. 2: Time evolution of the gravitational potential $\Phi(\eta)$ for the constant-w case. The three thin lines correspond to $k = \{0.01, 5, 500\}$ from top to bottom for the $w = -0.5$ case. The thick line is the scale-independent $w = -1$ case (ΛCDM).

IV. LINEAR $V(Q)$ CASE

Linear scalar field potential was looked by Andrei [32] in inflation theory long ago. Recently it is reviewed by Dimopoulos and Thomas [26] as a quintessence field within a quantum mechanically stable action. Due to the large Z-moduli of the wave function factor, the higher order potential derivatives d^nV/dQ^n ($n > 1$) can be neglected and only the linear term remains. After shifting the origin and considering the symmetry of $Q \leftrightarrow -Q$, the typical $V(Q)$ can be written as:

$$V(Q) = -V_1\, Q, \qquad V_1 = constant > 0. \tag{21}$$

This linear potential may eventually lead to negative total energy density and result in a rapid crunch. Let us consider the closed system defined in §2 to study the behavior of both background and perturbations for this model in detail. In order of magnitude, Eq.(2) gives $Q_0' \sim a_0^2\eta_0 V_1 \sim a_0 V_1/H_0$ and eq.(9) gives $Q_0'^2/2a^2 \sim |V_1 Q| \sim \langle\rho_Q\rangle \sim 3H_0^2\Omega_Q/8\pi G$ for $1+w \sim 1$. Combining these two, we will have $V_1 \sim H_0^2\sqrt{3\Omega_Q/8\pi G} \sim H_0^2 m_p$ ($\hbar = c = 1$) and $Q \sim \sqrt{3\Omega_Q/8\pi G} \sim m_p$. So this simple analysis sets the range of magnitude for the quintessence field quantities. Now we can redefine the quintessence field for later convenience:

$$V_1 = H_0^2\sqrt{\frac{3\Omega_Q}{8\pi G}}\,\tilde{V}_1, \quad Q = \sqrt{\frac{3\Omega_Q}{8\pi G}}\,\tilde{Q}, \quad \tilde{V}_1 \sim \tilde{Q} \sim 1, \;\; \tilde{V}_1 = constant > 0. \tag{22}$$

The closed system of equations can then be rewritten as:

$$\tilde{Q}_0'' + 2\mathcal{H}\tilde{Q}_0' - H_0^2 a_0^2 \bar{a}^2 \tilde{V}_1 = 0 \,, \tag{23}$$

$$\frac{\bar{a}'}{\bar{a}^2} = H_0 a_0 \sqrt{\frac{\Omega_m}{\bar{a}^3} + \Omega_Q \left(\frac{\tilde{Q}_0'^2}{2H_0^2 a_0^2 \bar{a}^2} - \tilde{V}_1 \tilde{Q}_0 \right)} \,, \tag{24}$$

$$\delta\tilde{Q}'' + 2\mathcal{H}\delta\tilde{Q}' + k^2\delta\tilde{Q} - 4\tilde{Q}_0'\Phi' - 2H_0^2 a_0^2 \bar{a}^2 \tilde{V}_1 \Phi = 0 \,, \tag{25}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = \frac{3}{2}\Omega_Q \left(-\tilde{Q}_0'^2 \Phi + \tilde{Q}_0' \delta\tilde{Q}' + H_0^2 a_0^2 \bar{a}^2 \tilde{V}_1 \delta\tilde{Q} \right) \,, \tag{26}$$

where the perturbation equations are in k-space. For the background, we take the initial condition as $\bar{a}(0) = 0$ and $\tilde{Q}_0'(0) = 0$ based on inflation theory. The $\tilde{Q}_0(0)$ value is chosen by requiring that today's quintessence energy density fits the observation, $\rho_Q(\eta_0) = 3H_0^2\Omega_Q/8\pi G$, or equivalently, $\tilde{Q}_0'^2/2H_0^2 a_0^2 - \tilde{V}_1\tilde{Q}_0(\eta_0) = 1$. Then this describes an initially static field rolling down a potential slope. Its effective EOS parameter evolves from -1 to some value $-1 < w(\eta_0) < 0$ today. Different slope values $\tilde{V}_1$ correspond to different $\tilde{Q}_0(0)$ and EOS parameter $w(\eta)$, but the same energy density $\rho_Q(\eta_0)$ today. The correspondence is listed in Table I. The detail has been worked out in the "doomsday" model [27]. For the $w(\eta_0) = -0.5$ case, the potential $V(Q)$ is shown in Figure 3.

TABLE I: Quintessence potential slope and starting position corresponding to $w(\eta_0)$

$\tilde{V}_1$	$\tilde{Q}_0(0)$	$w(\eta_0)$
2.738	-0.635	-0.5
2.424	-0.657	-0.6
2.077	-0.696	-0.7
1.677	-0.774	-0.8
1.172	-0.980	-0.9

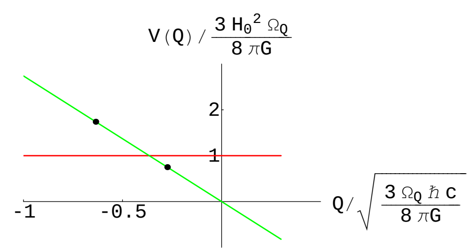

FIG. 3: Linear scalar field potential $V(Q)$ for the $w(\eta_0) = -0.5$ case shown as the slope. The two dots are the starting and today's positions of the field variable from left to right. The horizontal line is the $w = -1$ case (ΛCDM).

For the perturbations, again we take the initial condition as $\delta\tilde{Q}'(0) = \Phi'(0) = 0$ and $|\Phi(0)| \sim 10^{-5}$, based on inflation theory. Eq.(25) describes the oscillation of an intitially static quintessence perturbation field driven by the gravitational potential. At small scales ($k \gg 1$) the quintessence perturbations stay small. So the Φ evolution only depends on the background. The behavior of the perturbations is not sensitive to k or $\delta\tilde{Q}(0)$, as in the constant-w case. However, at scales comparable or bigger than the horizon size ($k \leq 1$), due to the zero quintessence mass ($V_{,QQ} = 0$) in the restoring term, the quintessence perturbation will stay near its initial value with a shift driven by the gravitational potential. Then the whole system becomes sensitive to the initial condition of quintessence perturbations, which is quite different from the constant-w case. For the $w(\eta_0) = -0.5$ special case, the dependence on k and $\delta\tilde{Q}(0)$ is indicated in Table II.

If we take the smooth initial condition $\delta\tilde{Q}(0) = 0$, then in terms of different k the perturbation evolution result is shown in Figure 4. Again we see that the scale dependence of perturbations evolutions is similar to the constant-w case in the last section. The quintessence perturbations stay suppressed, corresponding to more decay of the gravitational potential at small scales, but grow to the order of the gravitational potential accounting for less decay of the gravitational potential, with the transition scale around $k = 5$. Besides, the decay of the gravitational potential

TABLE II: Dependence on initial perturbations in the linear-$V(Q)$ case, for $w(\eta_0) = -0.5$

	$k = 0.1$		$k = 1$		$k = 10$	
$\delta\tilde{Q}(0)/\Phi(0)$	$\delta\tilde{Q}(\eta_0)/\Phi(0)$	$\Phi(\eta_0)/\Phi(0)$	$\delta\tilde{Q}(\eta_0)/\Phi(0)$	$\Phi(\eta_0)/\Phi(0)$	$\delta\tilde{Q}(\eta_0)/\Phi(0)$	$\Phi(\eta_0)/\Phi(0)$
10	12.077	3.862	9.411	3.262	0.147	0.741
1	1.752	1.116	1.469	1.054	0.108	0.730
0	0.605	0.811	0.586	0.808	0.104	0.729
-1	-0.542	0.506	-0.296	0.563	0.099	0.728
-10	-10.867	-2.240	-8.240	-1.646	0.060	0.717

is smaller than the constant-w case because its quintessence EOS parameter stays close to -1 at early times and only rises to $w(\eta_0)$ recently, so that the dark energy emerges later.

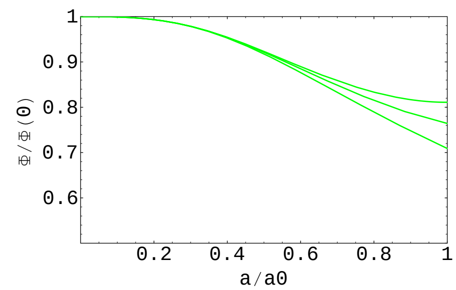

FIG. 4: Time evolution of the gravitational potential $\Phi(\eta)$ for the linear-$V(Q)$ case with $w(\eta_0) = -0.5$ and smooth initial condition. The three curves correspond to $k = \{0.01, 5, 500\}$ from top to bottom.

V. SUPERGRAVITY V(Q) CASE

The scalar field potential can also come from extended gauged supergravity, which has a close relation to M/string theory and extra dimensions. Masses of ultra-light scalars in these models [25] are quantized by the Hubble constant: $m^2 = n^2 H^2$. If the de-Sitter solution corresponds to a minimum of the effective potential, the universe eventually becomes de-Sitter space. If the de-Sitter solution corresponds to a maximum or a saddle point, which is the case in all known models based on $N = 8$ supergravity, the flat universe eventually stops accelerating and collapses to a singularity. For example, all known potentials of the gauged $N = 8$ supergravity have the universal feature $n^2 = \{-6, 4, 12\}$. Along the tachyon direction $n^2 = -6$, the scalar potential can be written as:

$$V(Q) = \frac{3H_0^2\Omega_Q\tilde{V}_0}{8\pi G}(1 - \tilde{Q}^2), \tag{27}$$

$$\tilde{V}_0 = constant > 0, \quad \tilde{Q} \equiv \frac{Q}{m_p}, \quad m_p \equiv \sqrt{\frac{\hbar c}{8\pi G}}$$

where $\tilde{V}_0$ and $\tilde{Q}$ are dimensionless and in the order of 1. Again we apply the closed system defined in §2 to this model. The background and perturbations equations are:

$$\tilde{Q}_0'' + 2\mathcal{H}\tilde{Q}_0' - 6H_0^2 a_0^2\Omega_Q\tilde{V}_0\bar{a}^2\tilde{Q}_0 = 0\ , \tag{28}$$

$$\frac{\bar{a}'}{\bar{a}^2} = \sqrt{\frac{H_0^2 a_0^2\Omega_m}{\bar{a}^3} + \frac{\tilde{Q}_0'^2}{6\bar{a}^2} + H_0^2 a_0^2\Omega_Q\tilde{V}_0\tilde{Q}_0(1 - \tilde{Q}_0^2)}\ , \tag{29}$$

$$\delta\tilde{Q}'' + 2\mathcal{H}\delta\tilde{Q}' + (k^2 - 6H_0^2 a_0^2\bar{a}^2\Omega_Q\tilde{V}_0)\delta\tilde{Q} - 4\tilde{Q}_0'\Phi' - 12H_0^2 a_0^2\bar{a}^2\Omega_Q\tilde{V}_0\tilde{Q}_0\Phi = 0\ , \tag{30}$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = -\frac{1}{2}\tilde{Q}_0'^2\Phi + \frac{1}{2}\tilde{Q}_0'\delta\tilde{Q}' + 3H_0^2 a_0^2\bar{a}^2\Omega_Q\tilde{V}_0\tilde{Q}_0\delta\tilde{Q}\ . \tag{31}$$

For the background, we again take the initial condition as $\bar{a}(0) = 0$ and $\tilde{Q}_0'(0) = 0$. We choose $\tilde{Q}_0(0) = \{0, 0.2, 0.3, 0.32\}$ and adjust $\tilde{V}_0$ so that today's quintessence energy density fits the observation, $\rho_Q(\eta_0) = 3H_0^2\Omega_Q/8\pi G$,

or equivalently, $\tilde{Q}_0'^2/6H_0^2a_0^2\Omega_Q + \tilde{V}_0[1-\tilde{Q}_0^2(\eta_0)] = 1$. Then this describes an initially static field rolling down a curved potential slope. Its effective EOS parameter evolves from -1 to some value $-1 < w(\eta_0) < 0$ today. Different initial positions $\tilde{Q}_0(0)$ correspond to different $\tilde{V}_0$ and EOS parameter $w(\eta)$, but the same energy density $\rho_Q(\eta_0)$ today. The correspondence is listed in Table III. For the $\tilde{Q}_0(0) = 0.3$ case, the potential $V(Q)$ is shown in Figure 5.

TABLE III: Quintessence initial position corresponding to $\tilde{V}_0$ and $w(\eta_0)$

$\tilde{Q}_0(0)$	$\tilde{V}_0$	$w(\eta_0)$
0	1	-1
0.2	1.124	-0.92
0.3	1.441	-0.67
0.32	1.645	-0.47

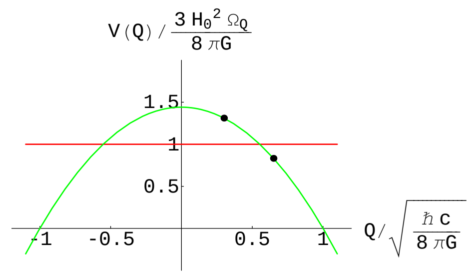

FIG. 5: Scalar field potential $V(Q)$ from supergravity for $\tilde{Q}_0(0) = 0.3$ shown as the curve. The two dots are the starting and today's positions of the field variable, from left to right. The horizontal line is the $w = -1$ case (ΛCDM).

The perturbations are analyzed as before. We take the initial condition as $\delta\tilde{Q}'(0) = \Phi'(0) = 0$ and $|\Phi(0)| \sim 10^{-5}$. Eq.(30) describes the oscillation of an initially static quintessence perturbation field driven by the gravitational potential. At small scales ($k \gg 1$) any initial quintessence perturbations get damped to be small, and the Φ evolution only depends on the background. This system is not sensitive to $\delta\tilde{Q}(0)$, as in the linear-$V(Q)$ case. At scales comparable to or bigger than the horizon size ($k \leq 1$), due to the tachyon quintessence mass ($V_{,QQ} < 0$) in the restoring term, a quintessence perturbation will stay near its initial value and even grow, with a shift driven by the gravitational potential. Then this system is more sensitive to the initial quintessence perturbations than the linear-$V(Q)$ case. For the $\tilde{Q}_0(0) = 0.3$ case, the dependence on k and $\delta\tilde{Q}(0)$ is indicated in Table IV.

TABLE IV: Dependence on initial perturbations in Supergravity-$V(Q)$ case, for $\tilde{Q}_0(0) = 0.3$

	$k = 0.1$		$k = 1$		$k = 10$	
$\delta\tilde{Q}(0)/\Phi(0)$	$\delta\tilde{Q}(\eta_0)/\Phi(0)$	$\Phi(\eta_0)/\Phi(0)$	$\delta\tilde{Q}(\eta_0)/\Phi(0)$	$\Phi(\eta_0)/\Phi(0)$	$\delta\tilde{Q}(\eta_0)/\Phi(0)$	$\Phi(\eta_0)/\Phi(0)$
10	23.412	3.016	18.640	2.589	0.207	0.753
1	3.018	1.011	2.521	0.967	0.159	0.746
0	0.752	0.788	0.731	0.787	0.153	0.746
-1	-1.514	0.565	-1.060	0.606	0.148	0.745
-10	-21.908	-1.440	-17.179	-1.016	0.100	0.739

This potential is unbounded from below and the theory is unstable. Since the potential can remain positive for $|\Phi| < 1$ for a time longer than the present age of the universe before it finally collapses, this model is sufficient to describe the background evolution of the universe. However, the tachyon mass makes the quintessence instability develop and affect the gravitational potential, and thus the system becomes sensitive to the initial conditions at large scales ($k < 10$). This puts more constraints on the initial quintessence conditions and this supergravity model itself. For the smooth initial condition $\delta\tilde{Q}(0) = 0$, this model has a similar scale-dependent perturbation evolution as the linear-$V(Q)$ model in last section. For $\tilde{Q}_0(0) = 0.3$ and $k = \{0.01, 5, 500\}$, the time evolution $\Phi(\eta)$ is shown in Figure 6.

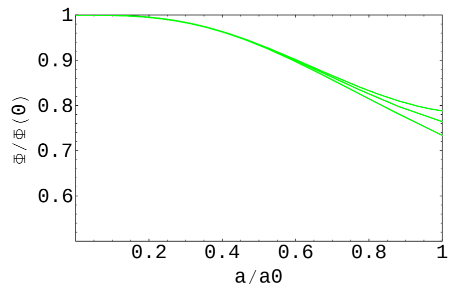

FIG. 6: Time evolution of the gravitational potential $\Phi(\eta)$ for the supergravity-$V(Q)$ case, with $\tilde{Q}_0(0) = 0.3$ and smooth initial condition. The three curves correspond to $k = \{0.01, 5, 500\}$ from top to bottom.

VI. DISCUSSION

In another viewpoint, we can apply the first-order matrix formulation in [22] to analyze the different quintessence perturbation behaviors in above $QCDM$ models. At large scales ($k \ll 1$), the relative entropy perturbation S and intrinsic entropy perturbation Γ [17, 18, 19] have coupled evolution equation

$$\frac{\partial}{\partial \ln(a/a_0)} \begin{pmatrix} S \\ \Gamma \end{pmatrix} = 3 \begin{pmatrix} w_Q - w_f + \gamma_f \Omega_f (w_f - c_{sQ}^2)/\gamma & \gamma_f \Omega_f (1 - c_{sQ}^2)/\gamma \\ -\gamma/2 & w_Q - \gamma/2 \end{pmatrix} \times \begin{pmatrix} S \\ \Gamma \end{pmatrix}$$

where $_f$ means the perfect fluid component, $\gamma_i \equiv 1 + w_i$, and $c_{sQ}^2 \equiv \dot{p}/\dot{\rho}$. For most of the time in our $QCDM$ models, CDM dominates and have $w_f \simeq 0$, $\Omega_f \simeq \gamma_f \simeq \gamma \simeq 1$, and $-1 < w_Q < -1/3$. Then the eigenvalues are $n_\pm = \frac{3}{2}\left[2w_Q - c_{sQ}^2 - \frac{1}{2} \pm \sqrt{(c_{sQ}^2 + \frac{1}{2})^2 - 2}\right]$. For the constant w case, $w_Q = c_{sQ}^2$ leads to $Re(n_\pm) = \frac{3}{2}(w_Q - 1/2) < 0$, corresponding to decaying entropy perturbations. While for the linear $V(Q)$ case, $c_{sQ}^2 \simeq -2$ leads to $n_\pm = (3w_Q + 3, 3w_Q + \frac{3}{2})$, corresponding to at least one growing entropy perturbation mode. This fits our previous analysis well.

In all $QCDM$ models that we studied above, the quintessence perturbations have the common characteristic feature of scale dependence. At small scales the perturbations do not grow and the gravitational potential decays more only due to background effects; at large scales the perturbations grow and sustain the gravitational potential. On the other hand, because of the different quintessence potentials, the three $QCDM$ models have different sensitivity to the initial conditions on quintessence perturbations. The constant-w model has a large quintessence mass at early times and damps perturbations fast so that it is insensitive to initial conditions at all scales. In this model the quintessence has no effect before the last-scattering time. In contrast, the linear-$V(Q)$ model and supergravity-$V(Q)$ model have zero and tachyon quintessence mass respectively, and their perturbations remain near their initial value ($\delta\tilde{Q}(0)/\Phi(0) \le 10^4$) at large scales. Since the EOS parameter w in these two models approaches -1 at early times, their dark energy dominates only very recently, like the ΛCDM model. So the total back-reaction of dark energy perturbations (background times relative perturbations) on other perturbations remains negligible before the last-scattering time, when the universe evolves like $a \propto \eta^2$, $Q_0 \simeq Q_0(0)$, $Q_0' \simeq 0$, $\delta Q \simeq \delta Q(0)$, and $\Phi \simeq \Phi(0)$. So they do not change the primary CMB anisotropies either. But later when the dark energy background become dominant, this back-reaction of dark energy perturbations will affect other perturbations drastically, depending on how smooth the initial perturbations are.

For smooth initial conditions in all models the dark energy has no primary CMB effect except that its background contributes to the distance to the last-scattering surface. But later on both the dark energy background and perturbations will contribute to the change of the gravitational potential, which may lead to a considerable secondary CMB effect such as the ISW effect [30]. For the scale-invariant primordial spectrum $P_\Phi \equiv |\Phi(0,k)|^2 = A^2/k^3$, the large-scale CMB power spectrum due to the SW effect is:

$$C_l^{SW} = \frac{A^2}{36\pi^2 l(l+1)} K_l^2 \,,$$
$$K_l^2 = 2l(l+1) \int_0^\infty \frac{dk}{k} \left[j_l(k\eta_0) + 6 \int_{\eta_{LS}}^{\eta_0} d\eta f'(\eta, k) j_l(k(\eta_0 - \eta)) \right]^2 \,, \tag{32}$$

where $j_l(z)$ is the spherical Bessel function and $f(\eta, k)$ is the evolution factor of the gravitational potential defined as $\Phi(\eta, k) \equiv \Phi(0, k) f(\eta, k)$. The integral over the $j_l^2(k\eta_0)$ term corresponds to the primary SW effect on the last-scattering surface, and the integral over the remaining terms corresponds to the ISW effect. The flat CDM model

gives $f' = 0$ and only the primary SW effect, with $K_l^2 = 1$. The flat ΛCDM model has no dark energy perturbations and gives a scale-independent evolution factor $f(\eta)$ which leads to a positive ISW effect in the low-l CMB power spectrum [31], rising at the lower-l ($l = 2, 3$) end. For the flat $QCDM$ model, the bigger EOS parameter w (≥ -1) brings out the dark energy earlier and decreases the gravitational potential more, corresponding to a bigger f' value for all scales and affecting the magnitude of the CMB power spectrum. The quintessence perturbations grow differently at different scales and give a k-dependent $|f'(\eta, k)|$, which is bigger at small scales than at large scales with the transition around the horizon size. This means that the lower-l ($l = 2, 3$) multipoles will get suppressed relative to the higher-l ones such that the whole low-l spectrum will get flattened. These two effects can be seen by numerical calculation for the standard ΛCDM model, constant-w $QCDM$ model with $w = -0.5$, and linear-$V(Q)$ $QCDM$ model with $w(\eta_0) = -0.5$ in Figure 7. This is done with eq.(32) approximately for illustrative purpose. Exact results should refer to more accurate numerical work like the CMB-FAST code. For non-smooth initial conditions, the perturbation evolution can be seen in Table II and Table IV, and corresponds to very unusual cosmological scenarios which we will not discuss further.

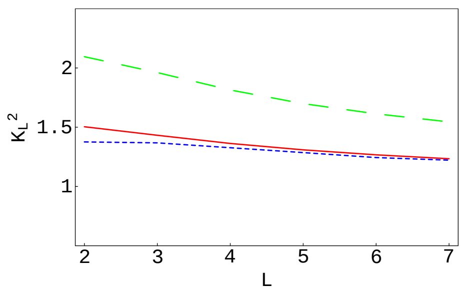

FIG. 7: The low-l CMB power spectrum due to the Sachs-Wolfe effect in various $QCDM$ models with the smooth initial condition. The solid line is for the ΛCDM model, the dash line for the constant-w $QCDM$ model with $w = -0.6$, and the dot line for the linear-$V(Q)$ $QCDM$ model with $w(\eta_0) = -0.5$.

The scale dependence and sensitivity to initial perturbations can also be tested in other ways. The CMB results give the primordial Φ_k spectrum, while the gravitational lensing and other large-scale observations yield Φ_k information at low redshift. The ratio of these two, the evolution of gravitational potential, can be directly compared with the theoretical plot on different scales shown above for various $QCDM$ models. Furthermore, since the matter density perturbation is related to the gravitational potential by eq.(5), the evolution of gravitational potential can also be inferred from the measurement of the visible matter density spectrum (by SDSS, etc.), and checked with $QCDM$ predictions.

By considering the dark energy perturbations in these $QCDM$ models, we see that they all have a scale dependence which is different from the ΛCMD model, and they have different sensitivity to initial conditions. These properties make the dark energy perturbations an effective way to test these models. More precise cosmological observations in the future will give better constraints on these $QCDM$ models.

VII. ACKNOWLEDGEMENTS

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